Real Analysis HW7 Solution

39. If $f_n \to f$ almost uniformly, then $f_n \to f$ a.e and in measure

① Since $f_n \to f$ almost uniformly, $\forall n \in N_+$, $\exists E_n$

s.t $\mu(E_n) < \frac{1}{n^2}$, $f_n \rightrightarrows f$ on E_n^c uniformly

$\therefore$ $f_n \to f$ poitwisely on E_n^c

Consider $E = \limsup_{n\to\infty} E_n = \bigcap_{k=1}^{\infty}\left(\bigcup_{n\geq k} E_n\right)$ Since $\sum_{n=1}^{\infty} \mu(E_n) < \infty$

$\mu(E) = 0$

and $f_n \to f$ poitwisely on E^c $\therefore f_n \to f$ a.e

②

fix $\delta > 0$, we want to show $\mu(x: |f_n - f| > \delta) \to 0$ for Converge in measure

$\forall \varepsilon > 0$, find E, s.t $\mu(E) < \varepsilon$, and $f_n \rightrightarrows f$ on E^c

then we can find sufficiently large N, s.t $|f_n - f| < \delta$ for $n > N$.

$\therefore$ $\mu(x: |f_n - f| > \delta) \leq \mu(E) < \varepsilon$ for $n > N$

$\therefore$ $\mu(x: |f_n - f| > \delta) \to 0$ $\therefore$ $f_n \to f$ in measure.

40. Replace hypothesis "$\mu(x) < \infty$" by "$|f_n| \leq g$, $g \in L^1$"

we only need to show that $\mu(E_1(k)) < \infty$ under the new hypothesis. then $\mu(E_n(k)) \to 0$ by Continuity from above, and all other proofs remain the same

$2|g(x)| \geq |f_m(x) - f(x)| \geq \frac{1}{k}$

$E_1(k) = \bigcup_{m\geq 1} \{x: |f_m(x) - f(x)| \geq \frac{1}{k}\}$

$\subseteq \{x: 2|g(x)| \geq \frac{1}{k}\}$

$\therefore \mu(E_1(k)) \leq \mu(x: 2|g(x)| \geq \frac{1}{k})$

$\leq 2k \cdot \|g\|_{L^1} < \infty$

45. If $(X_j, \mathcal{M}_j)$ is measurable space for $j=1,2,3$. then $\bigotimes_1^3 \mathcal{M}_j = (\mathcal{M}_1 \otimes \mathcal{M}_2) \otimes \mathcal{M}_3$. Moreover, if μ_j is σ-finite measure on $(X_j, \mathcal{M}_j)$ then $\mu_1 \times \mu_2 \times \mu_3 = (\mu_1 \times \mu_2) \times \mu_3$

$\bigotimes_1^3 \mathcal{M}_j$ is σ-algebra generated by $\{A \times B \times C : A \in \mathcal{M}_1, B \in \mathcal{M}_2, C \in \mathcal{M}_3\}$

$(\mathcal{M}_1 \otimes \mathcal{M}_2) \otimes \mathcal{M}_3$ is σ-algebra generated by $\{E \times C : E \in \mathcal{M}_1 \otimes \mathcal{M}_2, C \in \mathcal{M}_3\}$

apparently $A \times B \in \mathcal{M}_1 \otimes \mathcal{M}_2$ $\quad \therefore \bigotimes_1^3 \mathcal{M}_j \subseteq (\mathcal{M}_1 \otimes \mathcal{M}_2) \otimes \mathcal{M}_3$

Consider all subset X of $X_1 \times X_2$, such that $X \times C \in \bigotimes_1^3 \mathcal{M}_j$

$\mathcal{Y} = \{X \subseteq X_1 \times X_2 : X \times C \in \bigotimes_1^3 \mathcal{M}_j\}$ is a σ-algebra

(1) $X^c \times C = X_1 \times X_2 \times C - X \times C \in \bigotimes_1^3 \mathcal{M}_j$

(2) $\left(\bigcup_{k=1}^{\infty} X^k\right) \times C = \bigcup_{k=1}^{\infty} X^k \times C \in \bigotimes_1^3 \mathcal{M}_j$

Also $\mathcal{Y}$ contains subsets like $\{A \times B,\ A \in \mathcal{M}_1, B \in \mathcal{M}_2\}$

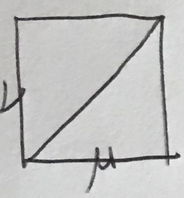
$\therefore \mathcal{Y} \supseteq \mathcal{M}_1 \otimes \mathcal{M}_2$ $\quad \therefore \bigotimes_1^3 \mathcal{M}_j \supseteq (\mathcal{M}_1 \otimes \mathcal{M}_2) \otimes \mathcal{M}_3$

$\therefore \bigotimes_1^3 \mathcal{M}_j = (\mathcal{M}_1 \otimes \mathcal{M}_2) \otimes \mathcal{M}_3$

$\mu_1 \times \mu_2 \times \mu_3 (A \times B \times C) = (\mu_1 \times \mu_2) \times \mu_3 (A \times B \times C) = \mu_1(A) \mu_2(B) \mu_3(C)$

Since μ_j is σ-finite, by unique extension, $\mu_1 \times \mu_2 \times \mu_3 = (\mu_1 \times \mu_2) \times \mu_3$

46. Let $X=Y=[0,1]$. $\mathcal{M}=\mathcal{N}=\mathcal{B}_{[0,1]}$. $\mu=$ Lebesgue measure. $\nu=$ counting measure. If $D=\{(x,x): x\in[0,1]\}$ is diagonal in $X\times Y$, then $\iint \chi_D \, d\mu \, d\nu$, $\iint \chi_D \, d\nu \, d\mu$, $\int \chi_D \, d(\mu\times\nu)$ are all unequal



$$\int_Y d\nu \int_X \chi_D(x,y)\, d\mu = \int_Y d\nu \int_X \chi_{\{y\}}(x)\, d\mu = \int_Y 0 \, d\nu = 0$$

$$\int_X d\mu \int_Y \chi_D(x,y)\, d\nu = \int_X d\mu \int_Y \chi_{\{x\}}(y)\, d\nu = \int_X 1 \cdot d\mu = 1$$

$$\int \chi_D \, d(\mu\times\nu) = \mu\times\nu(D)$$

$$= \inf\left\{ \sum_{n=1}^{\infty} \mu(A_n)\nu(B_n) : D \subseteq \bigcup_{n=1}^{\infty} A_n\times B_n,\ A_n, B_n \in \mathcal{B}_{[0,1]} \right\}$$

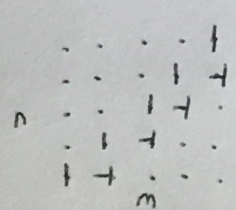
$D \subseteq \bigcup_{n=1}^{\infty} (A_n\cap B_n)\times(A_n\cap B_n) \subseteq \bigcup_{n=1}^{\infty} A_n\times B_n$, So we can further

Consider $\mu\times\nu(D) = \inf\left\{ \sum_{n=1}^{\infty} \mu(E_n)\nu(E_n) : D\subseteq \bigcup_{n=1}^{\infty} E_n\times E_n,\ E_n\in\mathcal{B}_{[0,1]} \right\}$

Since $\bigcup_{n=1}^{\infty} E_n = [0,1]$ we have $\sum_{n=1}^{\infty} \mu(E_n) \geq 1$

if $\mu(E_i)>0$, then $\nu(E_i)=\infty$ $\therefore \mu\times\nu(D)=\infty$

48. Let $X=Y=\mathbb{N}$, $\mathcal{M}=\mathcal{N}=\mathcal{P}(\mathbb{N})$. $\mu=\nu=$ counting measure. Define $f(m,n)=1$ if $m=n$, $f(m,n)=-1$ if $m=n+1$, and $f(m,n)=0$ otherwise. Then $\int |f| \, d(\mu\times\nu)=\infty$ and $\iint f\, d\mu\, d\nu$, $\iint f \, d\nu\, d\mu$ exist and are unequal.



$$|f| = \chi_{\{(m,n) \mid m=n \text{ or } m=n+1\}}$$

$$\int |f| \, d(\mu\times\nu) = (\mu\times\nu)\{(m,n) \mid m=n \text{ or } m=n+1\} = \infty$$

$\int d\mu \int f \, d\nu = 1$

Since for $m=0$, $\int f(0,n) d\nu = f(0,0) = 1$

$m \neq 0$, $\int f(m,n) d\nu = f(m,m) + f(m, m-1) = 0$

$\int d\nu \int f \, d\mu = 0$ Since for all n. $\int f(m,n) d\mu = f(n,n) + f(n+1, n)$
$= 0$

51. Let (X, M, μ). (Y, N, ν) be arbitrary measure spaces

a. if $f: X \mapsto C$ is M-measurable. $g: Y \to C$ is N-measurable, and $h(x,y) = f(x)g(y)$. then h is $M \otimes N$-measurable

b. if $f \in L^1(\mu)$, $g \in L^1(\nu)$. then $h \in L^1(\mu \times \nu)$ and

$$\int h \, d(\mu \times \nu) = \int f d\mu \cdot \int g \cdot d\nu$$

$f = \varphi_1 + i\varphi_2 \qquad g = \psi_1 + i\psi_2$

$$f(x)g(y) = (\varphi_1(x) + i\varphi_2(x)) \cdot (\psi_1(y) + i\psi_2(y))$$
$$= [\varphi_1(x)\psi_1(y) - \varphi_2(x)\psi_2(y)] + i[\varphi_1(x)\psi_2(y) + \varphi_2(x)\psi_1(y)]$$

So it's enough to consider real functions $f, g \in R$

$f = f^+ - f^- \qquad g = g^+ - g^-$

$$f(x)g(y) = (f^+(x) - f^-(x))(g^+(y) - g^-(y)) = f^+(x)g^+(y) + f^-(x)g^-(y) - f^+(x)g^-(y) - f^-(x)g^+(y)$$

So it's enough to consider nonnegative functions $f, g \geq 0$

$\forall a>0 \quad \{f(x)g(y)<a\} = \bigcup_{q\in\mathbb{Q}^+} \{f(x)<q\} \times \{g(y)<\frac{a}{q}\}$

Since f, g are measurable, and $\mathbb{Q}^+$ is countable

$\{f(x)g(y)<a\}$ is $\mathcal{M}\otimes\mathcal{N}$ - measurable $\therefore f(x)g(y)=h(x,y)$ is $\mathcal{M}\otimes\mathcal{N}$- measurable

For question b. by the same argument above. if it holds for nonnegative functions $f(x), g(y)$. then it's true for complex functions

$\exists$ Simple functions

$$0 \le \phi_1(x) \le \phi_2(x) \cdots \le \phi_n(x) \uparrow f(x)$$
$$0 \le \psi_1(y) \le \psi_2(y) \cdots \le \psi_n(y) \uparrow g(y)$$

$$\therefore 0 \le \phi_1(x)\psi_1(y) \le \phi_2(x)\psi_2(y) \cdots \le \phi_n(x)\psi_n(y) \uparrow f(x)g(y)$$

We show that $\int \phi_n(x)\psi_n(y)\, d(\mu\times\nu) = \int \phi_n(x)\, d\mu \cdot \int \psi_n(y)\, d\nu$

$\phi_n(x) = \sum_{i=1}^{s} a_i \chi_{E_i}(x) \quad \psi_n(y) = \sum_{j=1}^{t} b_j \chi_{F_j}(y) \quad \{E_i\} \quad \{F_j\}$ disjoint

$$\int \phi_n(x)\psi_n(y)\, d(\mu\times\nu) = \int \sum_{i,j} a_i b_j \cdot \chi_{E_i\times F_j}(x,y)\, d(\mu\times\nu)$$
$$= \sum_{i,j} a_i b_j (\mu\times\nu)(E_i\times F_j) = \sum_{i,j} a_i b_j \mu(E_i)\nu(F_j)$$
$$= \int \phi_n(x)\, d\mu \cdot \int \psi_n(y)\, d\nu$$

$$\therefore \int f(x)g(y)\, d(\mu\times\nu) = \lim_{n\to\infty} \int \phi_n(x)\psi_n(y)\, d(\mu\times\nu)$$
$$= \lim_{n\to\infty} \int \phi_n(x)\, d\mu \cdot \int \psi_n(y)\, d\nu = \int f(x)\, d\mu \cdot \int g(y)\, d\nu$$